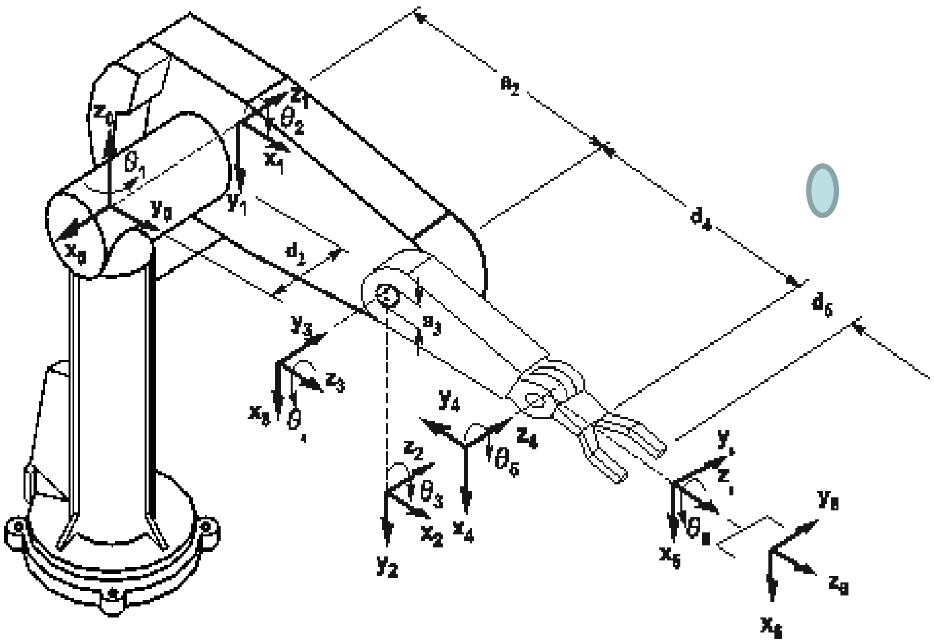
Robotics

Chapter 4
Inverse Manipulator Kinematics

4.1 INTRODUCTION

- Inverse Kinematics is the reverse of Forward Kinematics. (!)
- It is the calculation of joint values given the positions, orientations, and geometries of mechanism's parts

Given the numerical value of ${}^{0}_{N}T$, we attempt to find values of $\theta_1, \theta_2, \ldots, \theta_n$.



4.2 SOLVABILITY

Existence of solutions

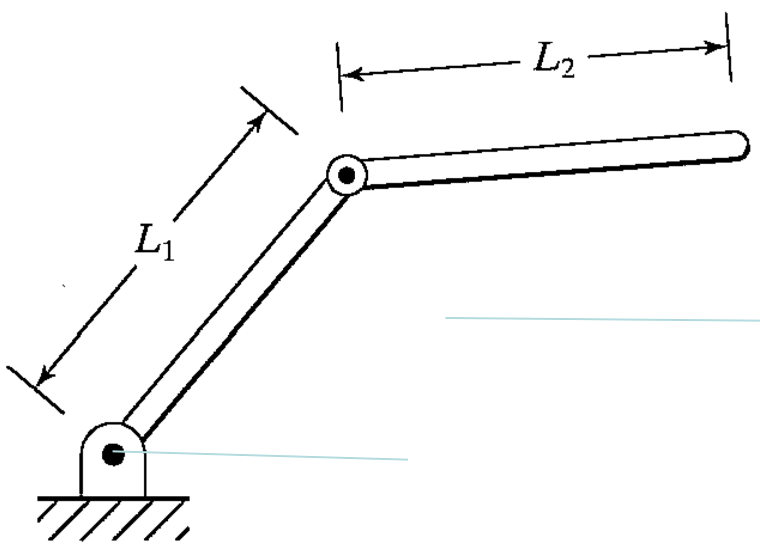
The question of whether any solution exists at all raises the question of the manipulator's **workspace**. Roughly speaking, workspace is that volume of space that the end-effector of the manipulator can reach. For a solution to exist, the specified goal point must lie within the workspace. Sometimes, it is useful to consider two definitions of workspace: **Dextrous workspace** is that volume of space that the robot end-effector can reach with all orientations. That is, at each point in the dextrous workspace, the end-effector can be arbitrarily oriented. The **reachable workspace** is that volume of space that the robot can reach in at least one orientation. Clearly, the dextrous workspace is a subset of the reachable workspace.

If $l_1 = l_2$:

If $l_1 \neq l_2$,

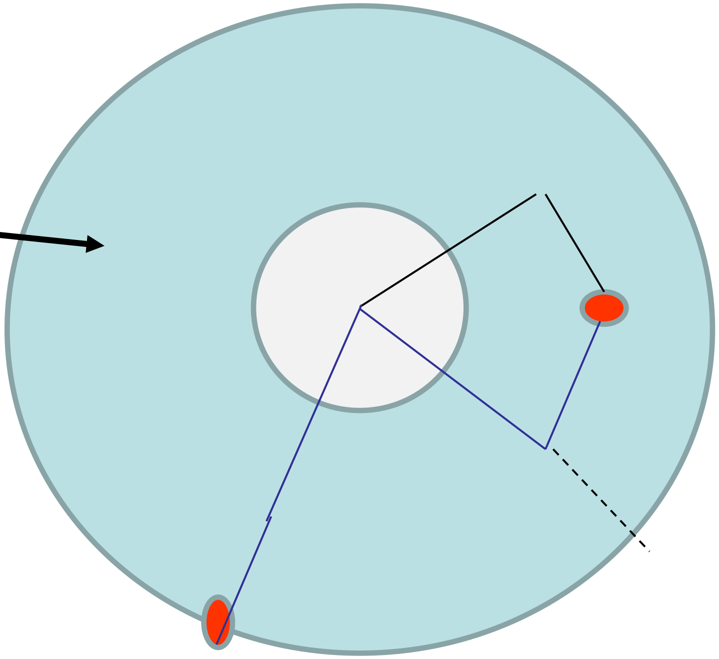
Consider the workspace of the two-link manipulator in Fig. 4.1. If $l_1 = l_2$, then the reachable workspace consists of a disc of radius $2l_1$. The dextrous workspace consists of only a single point, the origin. If $l_1 \neq l_2$, then there is no dextrous workspace, and the reachable workspace becomes a ring of outer radius $l_1 + l_2$ and inner radius $|l_1 - l_2|$. Inside the reachable workspace there are two possible orientations of the end-effector. On the boundaries of the workspace there is only one possible orientation.

$$0 \leq \theta_1 \leq 360, \quad 0 \leq \theta_2 \leq 360$$



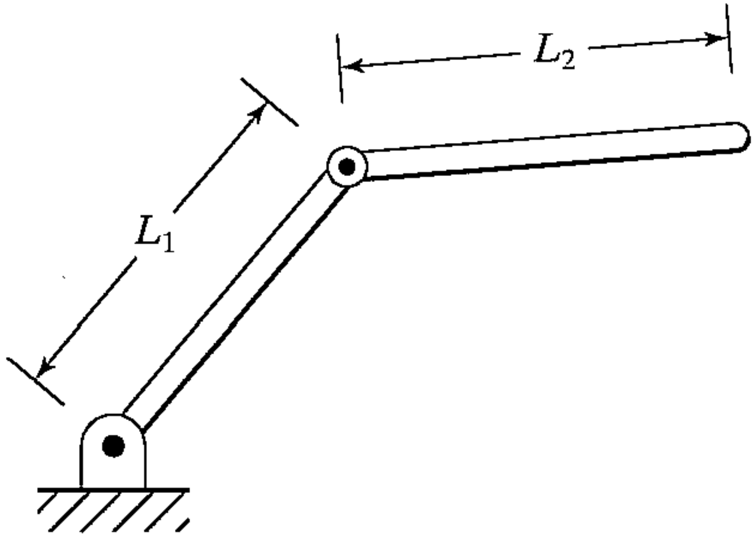
Reachable W.S: solid disk
Dexterous W.S: point

If $L_1 > L_2$



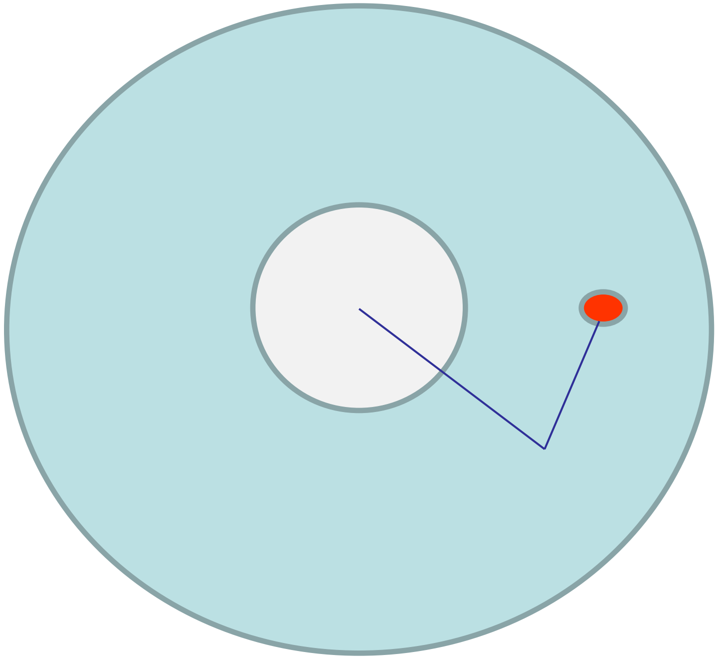
- Reachable Workspace
- Dextrous workspace (no dextrous)
- No of solutions (inner and boundary)

$$0 \le \theta_1 \le 360, \quad 0 \le \theta_2 \le 360$$

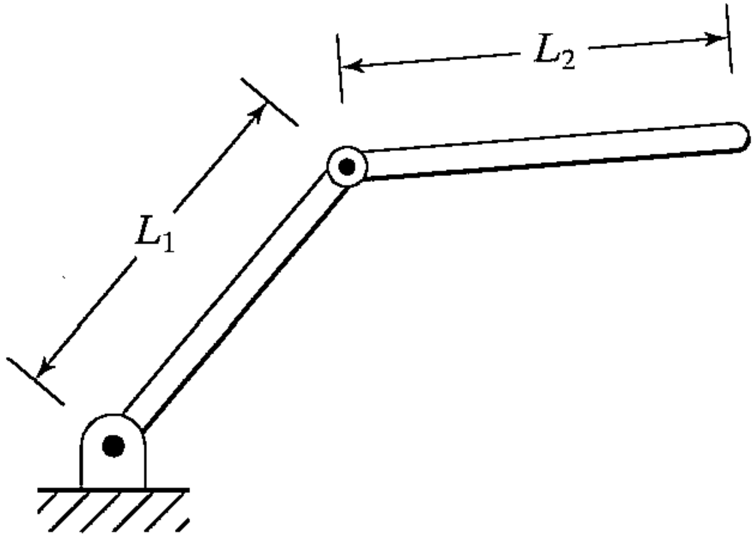


If $L_1 > L_2$

- Reachable Workspace
- Dextrous workspace
- No of solutions (inner and boundary)



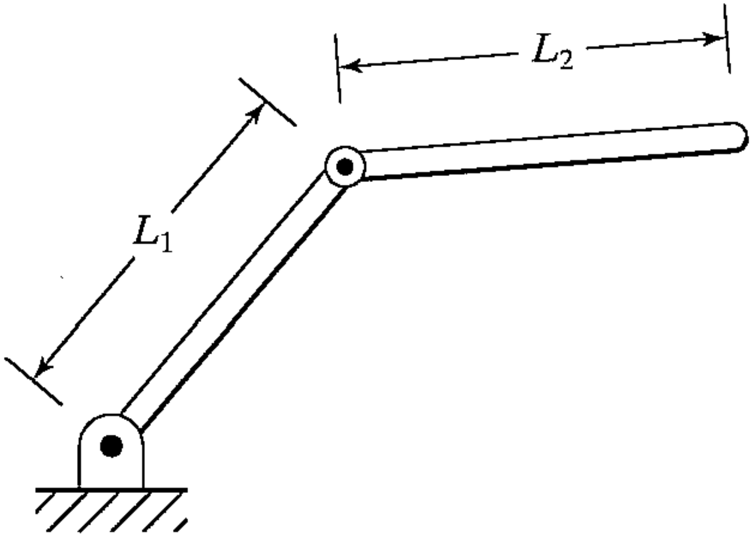
$$0 \le \theta_1 \le 360, \quad 0 \le \theta_2 \le 180$$



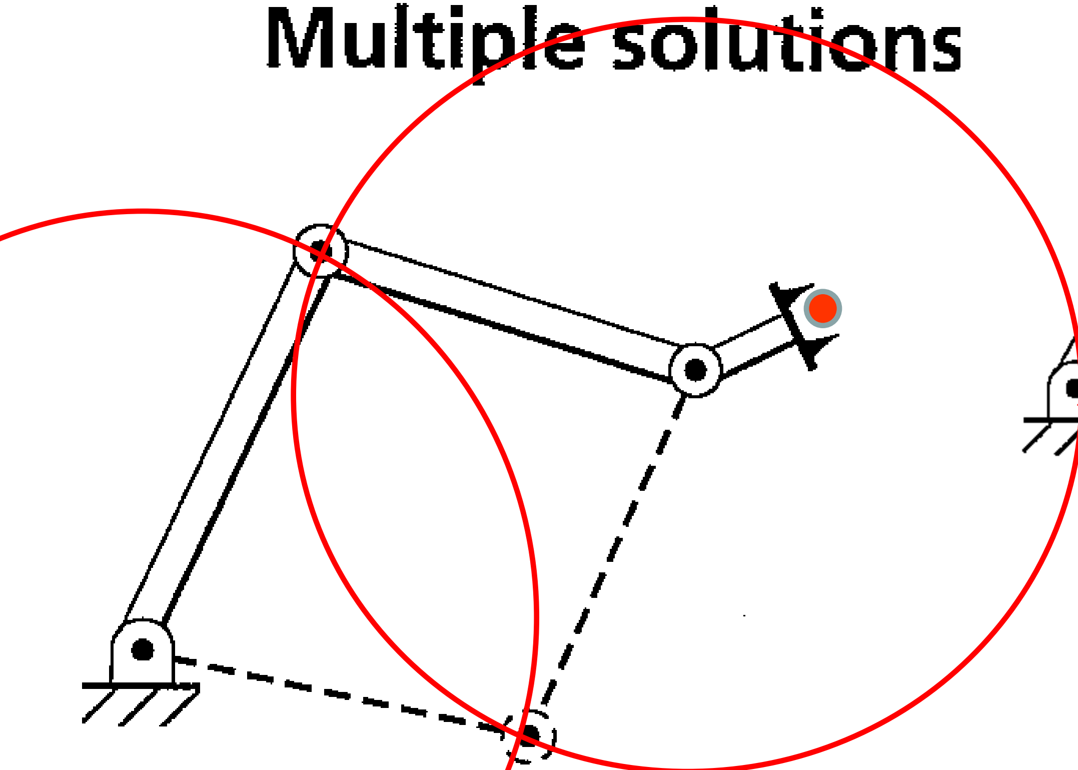
If $L_2 > L_1$

- Reachable Workspace
- Dextrous workspace
- No of solutions (inner and boundary)

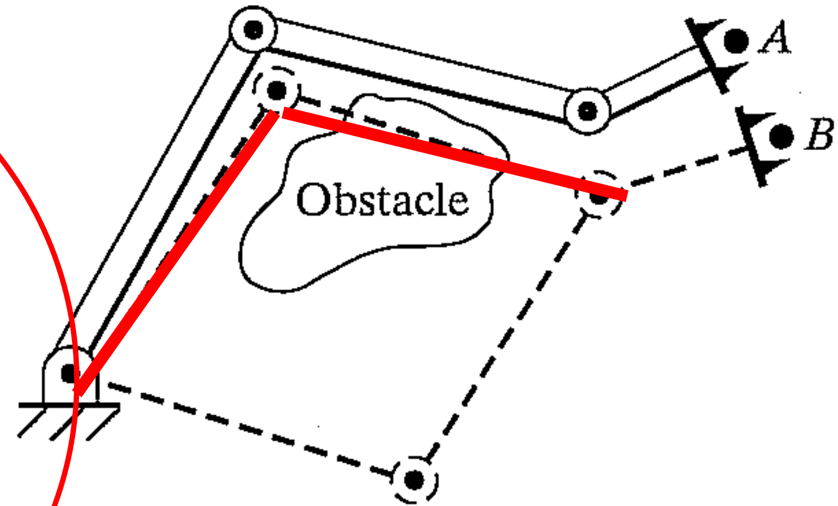
$$0 \le \theta_1 \le 360, \quad 0 \le \theta_2 \le 360$$



Multiple solutions



Dashed lines indicate a second solution.



closest solution.

. The presence of obstacles

moving smaller joints

PUMA 560 can reach certain goals with eight different solutions.

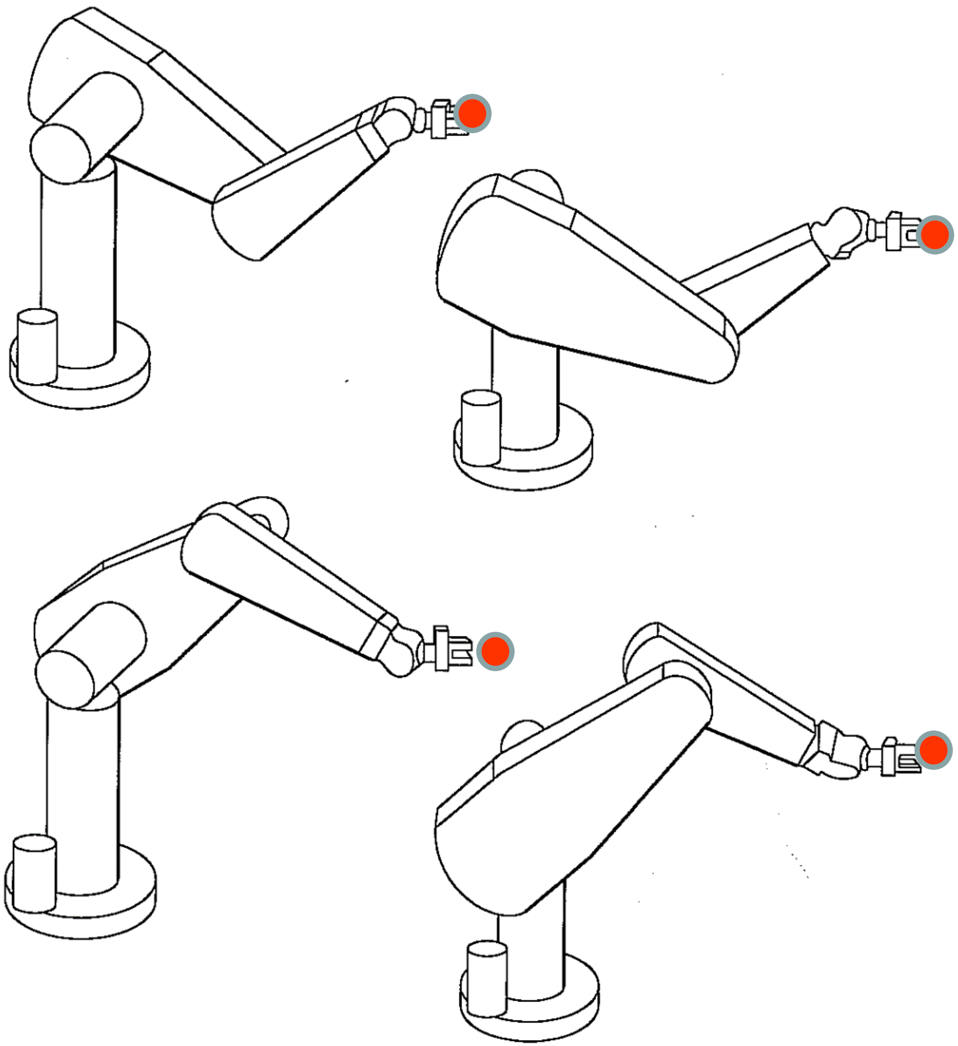


FIGURE 4.4: Four solutions of the PUMA 560.

$$\theta_4' = \theta_4 + 180°,$$
$$\theta_5' = -\theta_5,$$
$$\theta_6' = \theta_6 + 180°.$$

a_i	Number of solutions
$a_1 = a_3 = a_5 = 0$	$\leqslant 4$
$a_3 = a_5 = 0$	$\leqslant 8$
$a_3 = 0$	$\leqslant 16$
All $a_i \neq 0$	$\leqslant 16$

FIGURE 4.5: Number of solutions vs. nonzero a_i.

Method of solution

closed-form solutions and **numerical solutions**.

We will restrict our attention to closed-form solution methods.

"closed form" means a solution method based on analytic expressions

Within the class of closed-form solutions, we distinguish two methods of obtaining the solution: **algebraic** and **geometric**. These distinctions are somewhat hazy: Any geometric methods brought to bear are applied by means of algebraic expressions, so the two methods are similar. The methods differ perhaps in approach only.

A major recent result in kinematics is that, according to our definition of solvability, *all systems with revolute and prismatic joints having a total of six degrees of freedom in a single series chain are solvable*. However, this general solution is a numerical one. Only in special cases can robots with six degrees of freedom be solved analytically. These robots for which an analytic (or closed-form) solution exists are characterized either by having several intersecting joint axes or by having many α_i equal to 0 or ± 90 degrees. Calculating numerical solutions is generally time consuming relative to evaluating analytic expressions; hence, it is considered very important to design a manipulator so that a closed-form solution exists. Manipulator designers discovered this very soon, and now virtually all industrial manipulators are designed sufficiently simply that a closed-form solution can be developed.

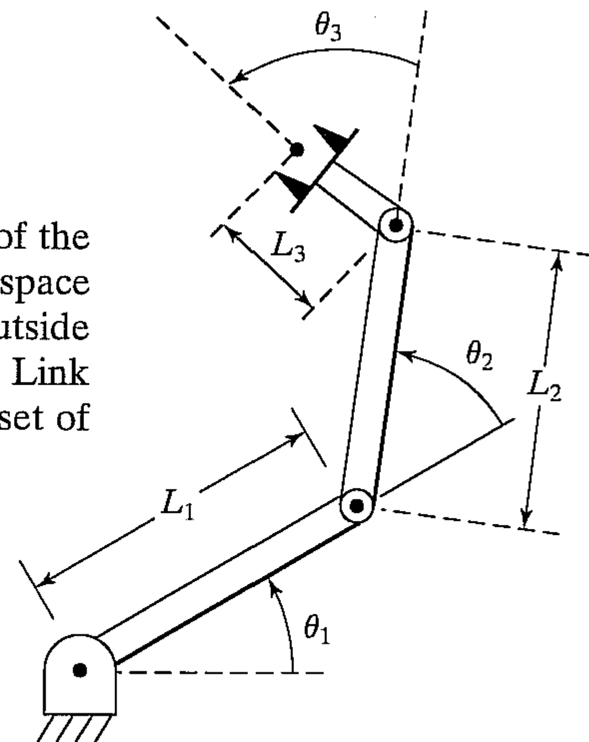
4.3 THE NOTION OF MANIPULATOR SUBSPACE WHEN $n < 6$

Give a description of the subspace of ${}^B_W T$ for the three-link manipulator

The subspace of ${}^B_W T$ is given by

$$
{}^B_W T = \begin{bmatrix} c_\phi & -s_\phi & 0.0 & x \\ s_\phi & c_\phi & 0.0 & y \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

where x and y give the position of the wrist and ϕ describes the orientation of the terminal link. As x, y, and ϕ are allowed to take on arbitrary values, the subspace is generated. Any wrist frame that does not have the structure of (4.2) lies outside the subspace (and therefore lies outside the workspace) of this manipulator. Link lengths and joint limits restrict the workspace of the manipulator to be a subset of this subspace.

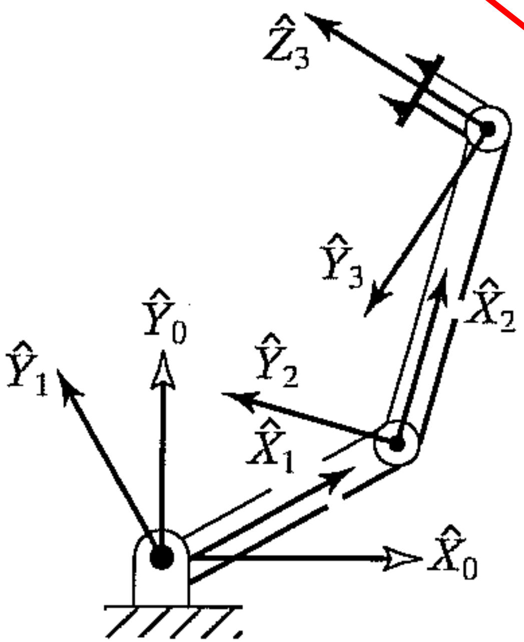


Algebraic solution

$$Find \; = \theta_1, \theta_2, \theta_3$$

given

$${}^0_3T = \begin{bmatrix} c_{123} & -s_{123} & 0.0 & l_1c_1 + l_2c_{12} \\ s_{123} & c_{123} & 0.0 & l_1s_1 + l_2s_{12} \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

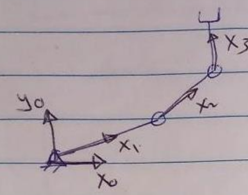


$${}^0_3T = \begin{bmatrix} c_\phi & -s_\phi & 0.0 & x \\ s_\phi & c_\phi & 0.0 & y \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Algebraic so

Inverse Kinematic (Algebric method).

$${}^{0}_{3}T = \begin{bmatrix} C_{123} & -S_{123} & 0 & L_1C_1+L_2C_{12} \\ S_{123} & C_{123} & 0 & L_1S_1+L_2S_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C\phi & -S\phi & 0 & X \\ S\phi & C\phi & 0 & Y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$X = L_1C_1 + L_2C_{12}$

$Y = L_1S_1 + L_2S_{12}$

But $C_1C_2 \mp S_1S_2 = C(1 \pm 2)$

$$X^2+Y^2 = (L_1^2C_1^2 + L_2^2C_{12}^2 + 2L_1C_1L_2C_{12}) + (L_1^2S_1^2 + L_2^2S_{12}^2 + 2L_1S_1L_2S_{12})$$

$$X^2+Y^2 = L_1^2 + L_2^2 + 2L_1L_2(C_1C_{12} + S_1S_{12})$$

$$X^2+Y^2 = L_1^2 + L_2^2 + 2L_1L_2\, C(\theta_1 - (\theta_1+\theta_2)) = L_1^2 + L_2^2 + 2L_1L_2C_2$$

$$\therefore C_2 = \frac{X^2+Y^2-L_1^2-L_2^2}{2L_1L_2}$$

$-1 \le C_2 \le 1$ solution exist. otherwise point out of workspace.

$$S_2 = \mp\sqrt{1-C_2^2}$$

$$\theta_2 = \text{Atan2}(S_2, C_2)$$

$$X = L_1C_1 + L_2(C_1C_2 - S_1S_2) = (L_1+L_2C_2)C_1 - (L_2S_2)S_1$$

$$y = L_1S_1 + L_2(C_1S_2 + S_1C_2) = (L_1+L_2C_2)S_1 + (L_2S_2)C_1$$

$K_1 = L_1 + L_2C_2 \qquad K_2 = L_2S_2$

$\therefore X = K_1C_1 - K_2S_1$

$\left[y = K_2C_1 + K_1S_1\right]\frac{K_2}{K_1}$

$$X + \frac{K_2}{K_1}y = \left(K_1 + \frac{K_2^2}{K_1}\right)C_1 \Rightarrow \frac{XK_1+K_2y}{K_1} = \left(\frac{K_1^2+K_2^2}{K_1}\right)C_1$$

$$C_1 = \left(\frac{K_1}{K_1^2+K_2^2}\right)\left(\frac{XK_1+K_2y}{K_1}\right) = \frac{XK_1+K_2y}{K_1^2+K_2^2}$$

Algebraic solu

$$K_1^2 + K_2^2 = (l_1 + l_2 C_2)^2 + (l_2 S_2)^2 = l_1^2 + l_2^2 C_2^2 + 2 l_1 l_2 C_2 + l_2^2 S_2^2$$

$$K_1^2 + K_2^2 = l_1^2 + l_2^2 + 2 l_1 l_2 C_2$$

$$C_1 = \frac{x K_1 + K_2 y}{K_1^2 + K_2^2} = \frac{(l_1 + l_2 C_2) x + (l_2 S_2) y}{l_1^2 + l_2^2 + 2 l_1 l_2 C_2}$$

Same way

$$S_1 = \frac{(l_1 + l_2 C_2) y - (l_2 S_2) x}{l_1^2 + l_2^2 + 2 l_1 l_2 C_2}$$

$$\theta_1 = A\tan 2(S_1, C_1)$$

$$\phi = A\tan 2(s\phi, c\phi) = A\tan 2(r_{21}, r_{11})$$

$$\theta_3 = \phi - \theta_1 - \theta_2$$

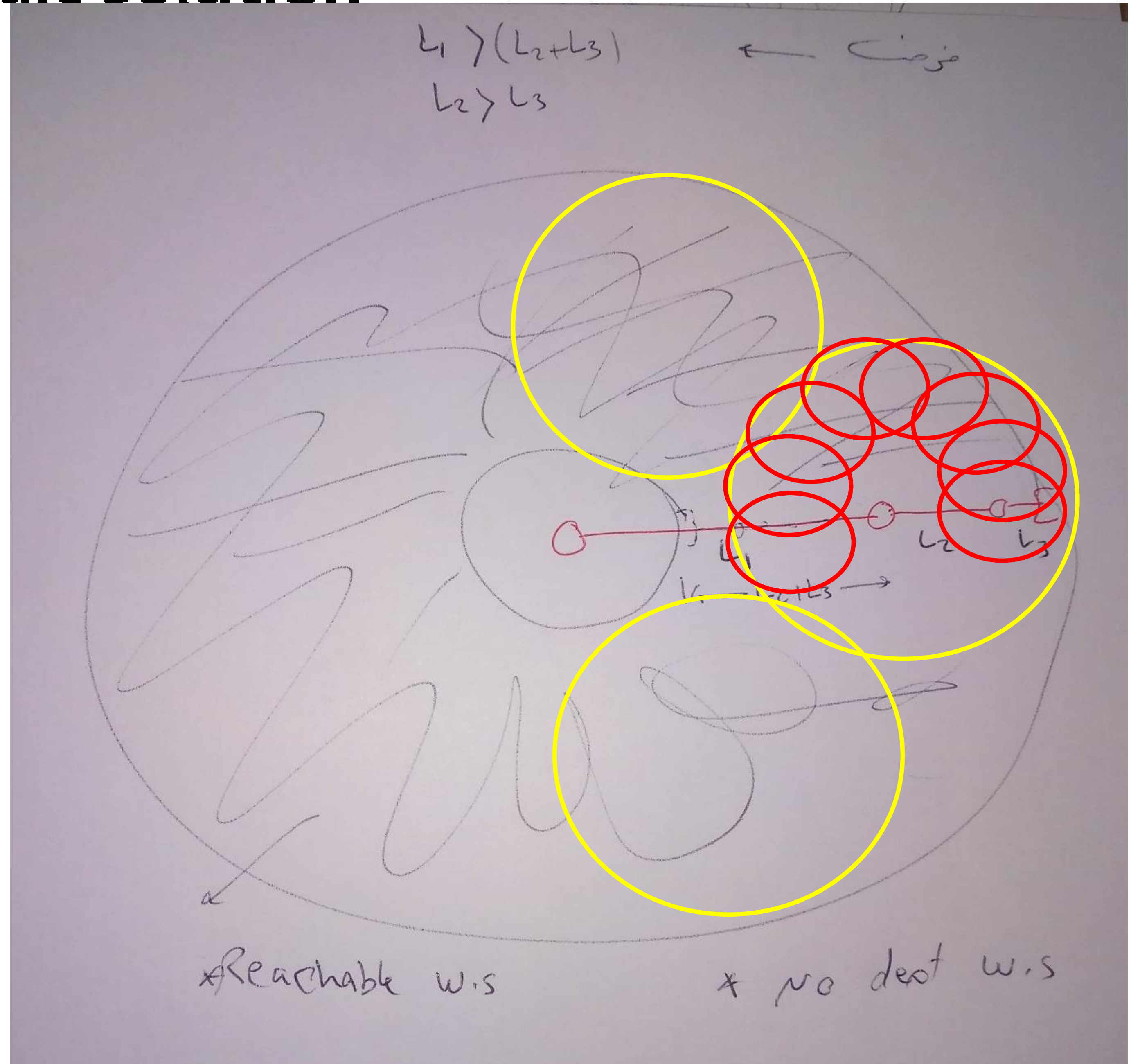
Appendix C-12

$$a \cos\theta - b \sin\theta = c$$

$$a \sin\theta + b \cos\theta = d$$

$$\theta = A\tan 2(ad - bc, ac + bd)$$

Algebraic solution



ALGEBRAIC SOLUTION BY REDUCTION TO POLYNOMIAL

Transcendental equations are often difficult to solve because, even when there is only one variable (say, θ), it generally appears as $\sin\theta$ and $\cos\theta$. Making the following substitutions, however, yields an expression in terms of a single variable, u:

$$u = \tan\frac{\theta}{2},$$
$$\cos\theta = \frac{1-u^2}{1+u^2}, \tag{4.35}$$
$$\sin\theta = \frac{2u}{1+u^2}.$$

EXAMPLE 4.3

Convert the transcendental equation

$$a\cos\theta + b\sin\theta = c$$

into a polynomial in the tangent of the half angle, and solve for θ.

Substituting from (4.35) and multiplying through by $1+u^2$, we have

$$a(1-u^2) + 2bu = c(1+u^2).$$

Collecting powers of u yields

$$(a+c)u^2 - 2bu + (c-a) = 0,$$

which is solved by the quadratic formula:

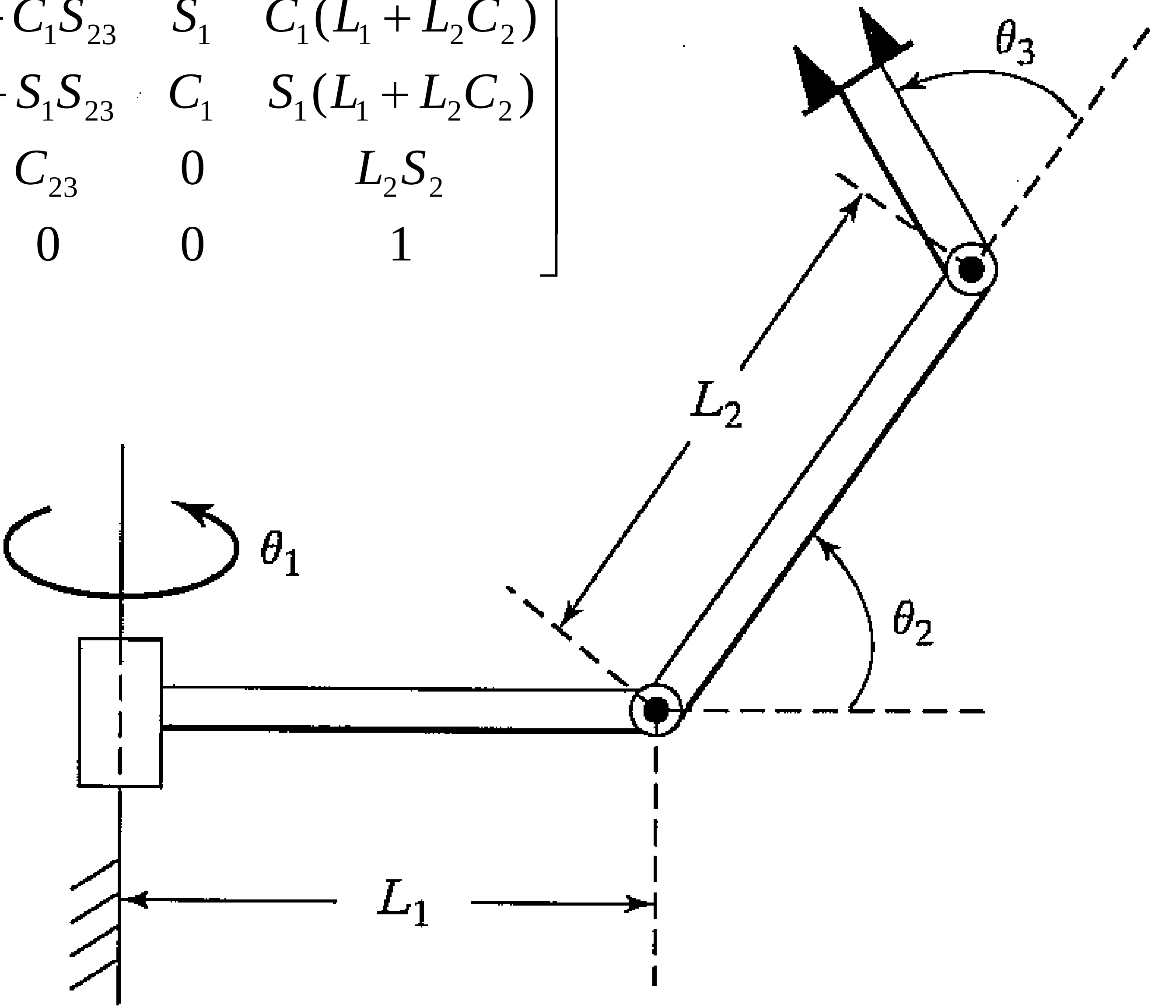
$$u = \frac{b \pm \sqrt{b^2+a^2-c^2}}{a+c}.$$

Hence,

$$\theta = 2\tan^{-1}\left(\frac{b \pm \sqrt{b^2+a^2-c^2}}{a+c}\right).$$

- Given the below transformation matrix solve inverse kinematic problem
- Sketch the workspace

$$ {}^0_3T = \begin{bmatrix} C_1C_{23} & -C_1S_{23} & S_1 & C_1(L_1+L_2C_2) \\ S_1S_{23} & -S_1S_{23} & C_1 & S_1(L_1+L_2C_2) \\ S_{23} & C_{23} & 0 & L_2S_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$



3	0	L_2	0	θ_3	2

$$^{0}_{3}T = \begin{bmatrix} C_1 C_{23} & -C_1 S_{23} & S_1 & (L_1 + L_2 C_2) C_1 \\ S_1 S_{23} & -S_1 S_{23} & +C_1 & (L_1 + L_2 C_2) S_1 \\ S_{23} & C_{23} & 0 & L_2 S_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\theta_1 = \text{Atan2}(r_{13}, r_{23}) \quad \text{or} \quad \text{Atan2}(P_y, P_x)$$

$$P_x^2 + P_y^2 = (L_1 + L_2 C_2)^2 C_1^2 + (L_1 + L_2 C_2)^2 S_1^2 = (L_1 + L_2 C_2)^2$$

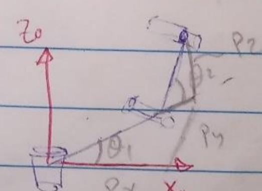
$$C_2 = \frac{\sqrt{P_x^2 + P_y^2} - L_1}{L_2}$$

$$S_2 = \frac{P_z}{L_2}$$

$$\Rightarrow \theta_2 = \text{Atan2}(S_2, C_2) = \text{Atan2}\left(P_z, \sqrt{P_x^2 + P_y^2} - L_1\right)$$

$$\theta_{23} = \text{Atan2}(r_{31}, r_{32})$$

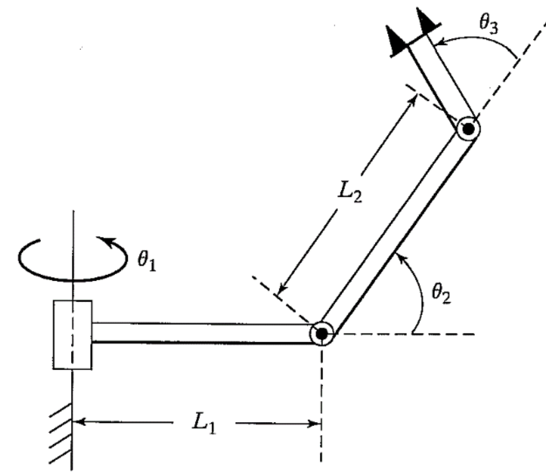
$$\theta_3 = \theta_{23} - \theta_2$$

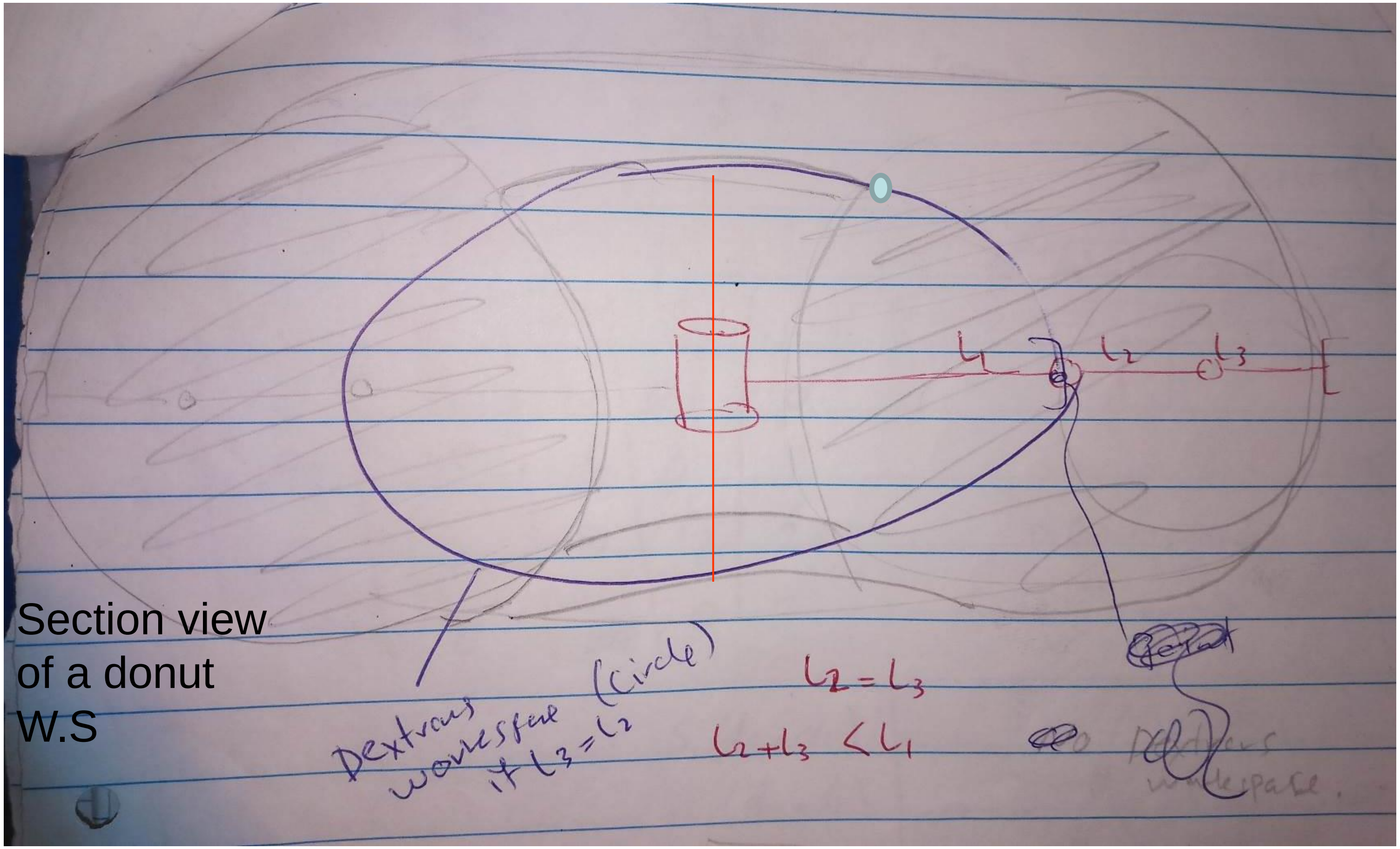


$$\theta_2 = \text{Atan2}\left(P_z, \sqrt{P_x^2 + P_y^2} - L_1\right)$$

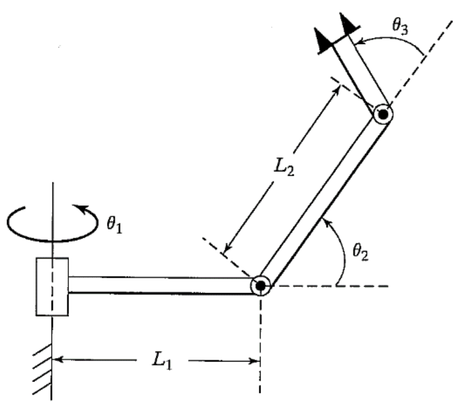
$$\theta_2 = \sin^{-1}\left(\frac{r_{34}}{L_2}\right)$$

$$\theta_1 = \text{Atan2}(P_y, P_x)$$

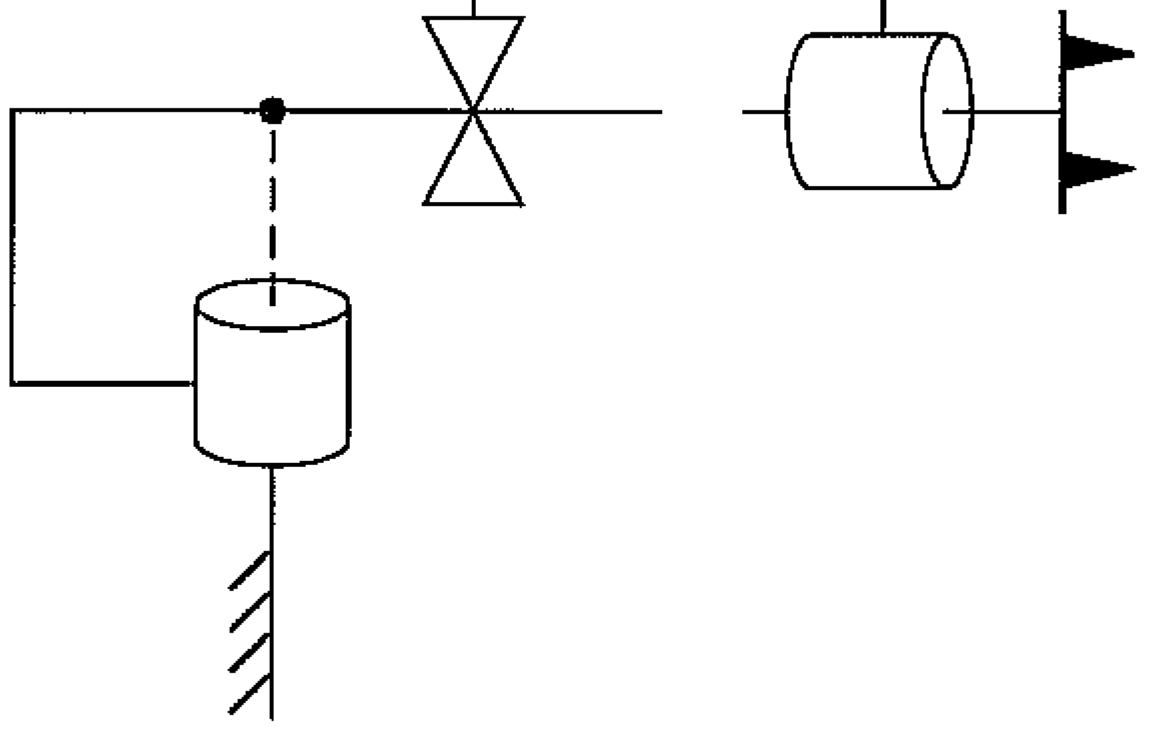




Section view of a donut W.S



- solve inverse kinematic problem
- Sketch the workspace



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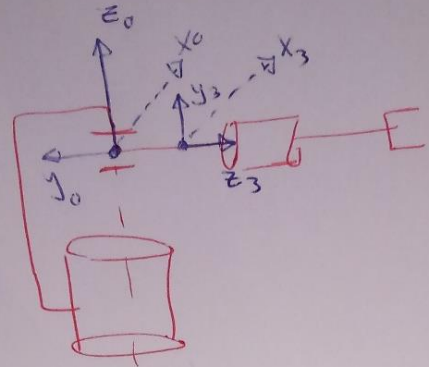


$$
{}^{0}_{3}T = \begin{bmatrix} C_1C_3 & -C_1S_3 & S_1 & S_1 d_2 \\ S_1C_3 & -S_1S_3 & -C_1 & -C_1 d_2 \\ S_3 & C_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

r31

r32

Algebric

$\theta_1 = Atan2(r_{13}, -r_{23})$ OR

OR

$\theta_1 = Atan2(r_{14}, -r_{24}) = Atan2(P_x, -P_y)$ (with $r_{14} \to P_x$, $-r_{24} \to -P_y$)

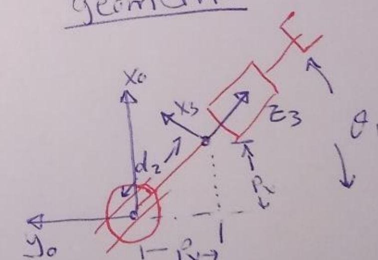
$S_1 d_2 = P_x \Rightarrow d_2 = \frac{P_x}{S_1}$

OR

$-C_1 d_2 = P_y \Rightarrow d_2 = \frac{P_y}{-C_1}$

$\theta_3 = Atan2(r_{31}, r_{32})$

geometric

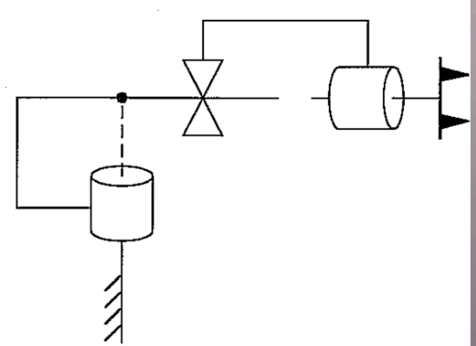


$\theta_1 = Atan2(P_x, -P_y)$

Note $S_1 = \frac{P_x}{d_2}$, $C_1 = \frac{-P_y}{d_2}$

$d_2 = \frac{P_x}{S_1}$ or $d_2 = \frac{-P_y}{C_1}$

OR $d_2 = \sqrt{P_x^2 + P_y^2}$



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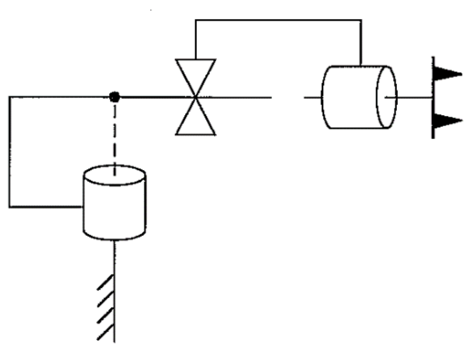

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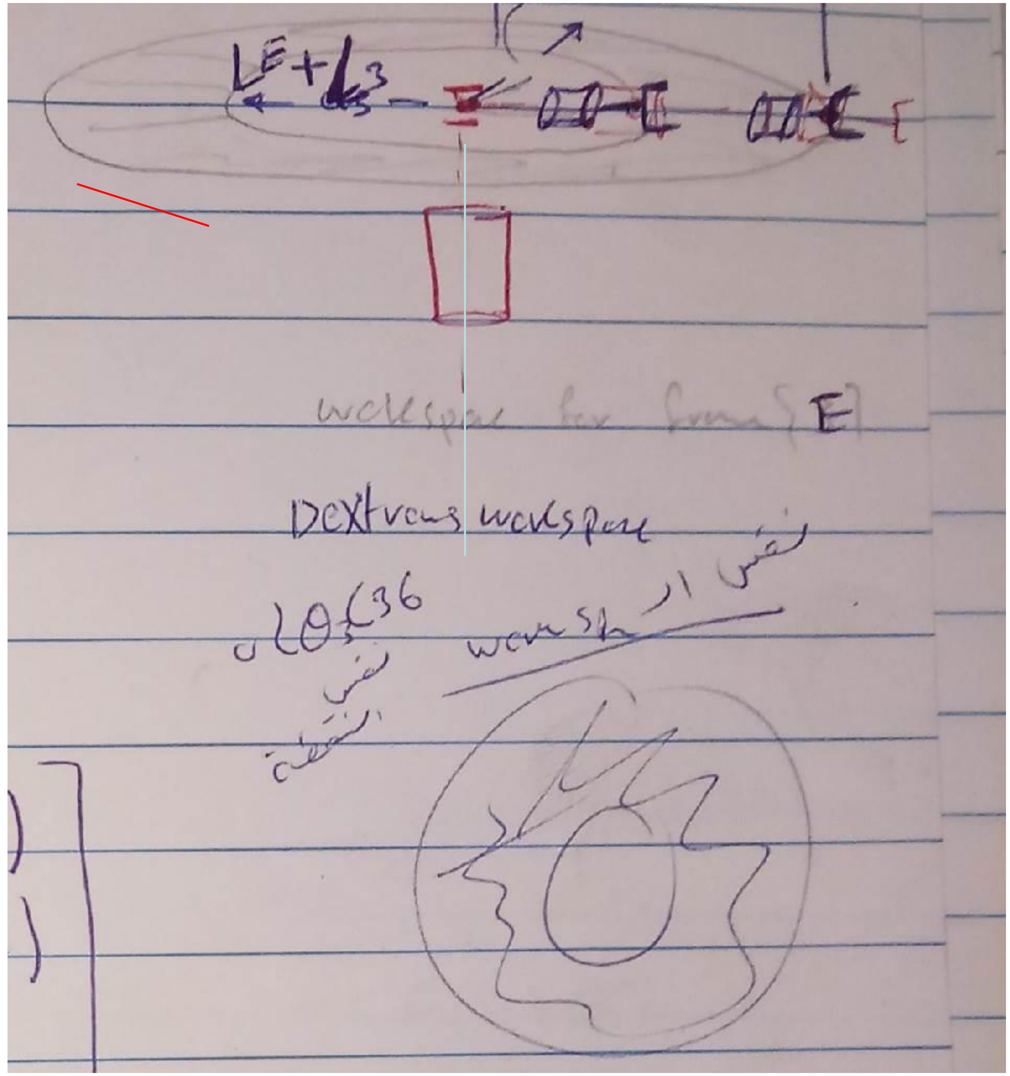


Problem 1(80 points):

For the RRP 3 DOF manipulator shown in figure 1, the homogeneous transformation matrix between the end-effector frame (3) and the base frame (0) is given as:

$$ {}^{0}_{3}T = \begin{bmatrix} C_1C_2 & -S_1 & C_1S_2 & C_1S_2d_3 - S_1 \\ S_1C_2 & C_1 & S_1S_2 & S_1S_2d_3 + C_1 \\ -S_2 & 0 & C_2 & C_2d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Find

θ_1, d_3, θ_2 ??

$0 \leq \theta_1 \leq 360 \quad 0 \leq \theta_2 \leq 360$

$0 \leq d_3 \leq \sqrt{3}$

$$ {}^{0}_{3}T = \begin{bmatrix} & ?? & & 0 \\ & & & 2 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

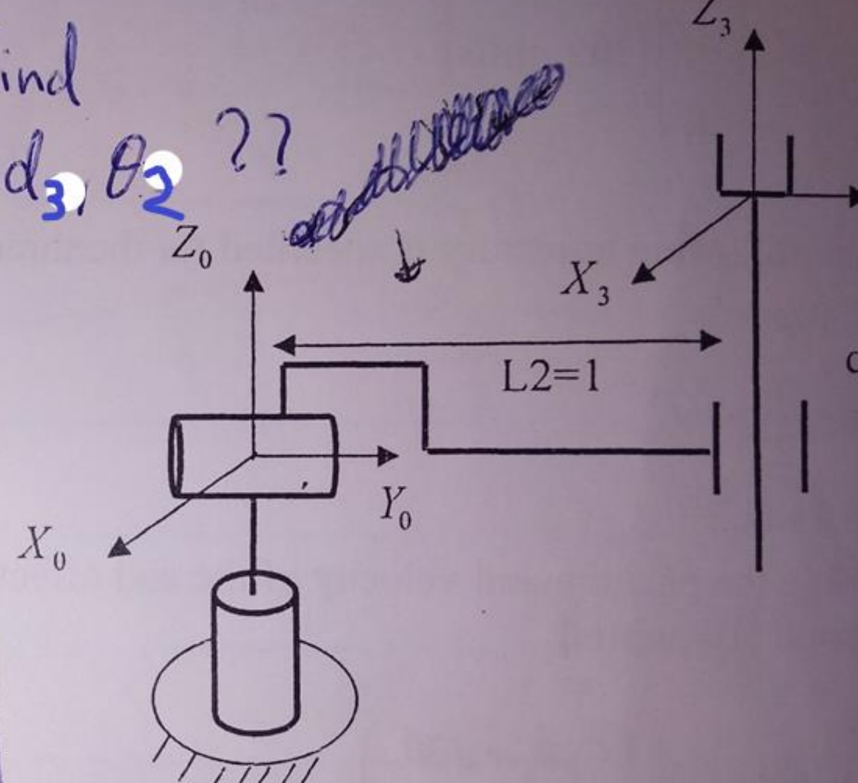


Fig 1: 3 DOF RRP manipulator

3 x 3 matrix

$P_x = C_1 S_2 d_3 - S_1$

$P_y = S_1 S_2 d_3 + C_1$

$P_z = C_2 d_3$

(5) $\Longleftrightarrow$

$0 = C_1 S_2 d_3 - S_1$ ... (1)

$2 = S_1 S_2 d_3 + C_1$ ... (2)

$0 = C_2 d_3$ ... (3)

(3)

From (3) $C_2 = 0$ or $d_3 = 0$

$\theta_2 = -\frac{\pi}{2}, \frac{\pi}{2}$ (5)

$\theta_2 = +\frac{\pi}{2}$ $S_2 = +1$, $C_2 = 0$

$\theta_2 = -\frac{\pi}{2}$ $S_2 = -1$, $C_2 = 0$

$$\begin{pmatrix} 0 = C_1 d_3 - S_1 \\ 2 = S_1 d_3 + C_1 \end{pmatrix}^2$$

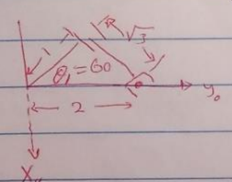
$0 = C_1^2 d_3^2 + S_1^2 - 2 S_1 C_1 d_3$

$4 = S_1^2 d_3^2 + C_1^2 + 2 S_1 C_1 d_3$ +

$4 = d_3^2 (C_1^2 + S_1^2) + (S_1^2 + C_1^2) + 0$

$4 - 1 = d_3^2 \Rightarrow d_3 = \pm\sqrt{3}$ (5)

(3)



$0 = C_1 d_3 - S_1$ (5)

$\frac{S_1}{C_1} = \frac{\sqrt{3}}{1} \Rightarrow \theta_1 = 60$

$$\begin{pmatrix} 0 = -C_1 d_3 - S_1 \\ 2 = -S_1 d_3 + C_1 \end{pmatrix}^2$$

$0 = C_1^2 d_3^2 + S_1^2 + 2 S_1 C_1 d_3$

$4 = C_1^2 + S_1^2 d_3^2 - 2 S_1 C_1 d_3$ +

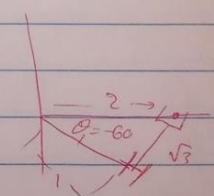
$4 = d_3^2 + 1$

$d_3 = \sqrt{3}$

$\frac{S_1}{C_1} = -d_3 = -\sqrt{3}$

$\theta_1 = -60$ (5)

(2)



If $d_3 = 0 \Rightarrow 0 = -S_1$, $2 = C_1$

$\hookrightarrow 2 \neq^{\text{never}} C_1$ not solution

$\therefore$ 2 Solution (2) (5)